

Filtering equation for a measurement of a coherent channel

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A stochastic model for a continuous photon counting and heterodyne measurement of a coherent source is proposed. A nonlinear filtering equation for the posterior state of a single-mode field in a cavity is derived by using the methods of quantum stochastic calculus. The posterior dynamics is found for the observation of a Bose field being initially in a coherent state. The filtering equations for counting and diffusion processes are given. © 2012 Optical Society of America

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1. Introduction

The quantum filtering equation describes the dynamics of a quantum system undergoing an indirect observation continuous in time [1–8]. In the model based on the quantum stochastic calculus (QSC) [9–12], worked out by Hudson and Parthasarathy, a role of measuring apparatus is played by a Bose field [13]. The dynamics of a compound system consisting of the quantum system in question and the Bose field is given by a unitary operator. The external field is here not only a source of a noise but it also drives the system and carries the information on it [10]. The time development of the posterior state conditioned by a trajectory of the results of continuous measurement of the output field is given by the stochastic and irreversible equation [5, 7, 8, 14–16].

In this paper we apply the theory of measurement continuous in time, based on QSC, to derive the quantum filtering equation for a single-mode field in a cavity interacting with the Bose field prepared in a coherent state. To get the posterior dynamics of the system we use the generating functional approach described in [1, 2], but, in contrast to the derivation given by

Belavkin when the Bose field is initially in the vacuum state, we consider the situation when the information about the system is extracted from a measurement of a coherent channel. We obtain the nonlinear filtering equation for a direct and heterodyne measurement of the signal escaping from cavity.

The statistics of the output process for the Bose field in a coherent state was considered by Barchielli in [17, 18], and for a mixture of coherent states by Barchielli and Pero in [19], but the estimation of the state of the indirectly observed quantum system was not given there.

2. Quantum stochastic calculus

In this section we present some basic ideas and notions of QSC in the boson Fock space [9, 11], which will be needed in our paper.

We model the electromagnetic field in the symmetric (Bose) Fock space \mathcal{F} over the Hilbert space $\mathcal{K} = \mathbb{C}^n \otimes L^2(\mathbb{R}_+)$ of all quadratically integrable functions from \mathbb{R}_+ into \mathbb{C}^n ,

$$\mathcal{F} = \mathbb{C} \oplus \left(\bigoplus_{k=1}^{\infty} \mathcal{K}^{\otimes_s k} \right). \quad (1)$$

The space \mathcal{K} called ‘one-particle space’ relates to the states of the photons. For every $f \in \mathcal{K}$ we define the exponential vector $e(f) \in \mathcal{F}$ by

$$e(f) = (1, f, (2!)^{-1/2} f \otimes f, (3!)^{-1/2} f \otimes f \otimes f, \dots). \quad (2)$$

The inner product of exponential vectors in \mathcal{F} takes the form

$$\langle e(g) | e(f) \rangle_{\mathcal{F}} = \exp \langle g | f \rangle_{\mathcal{K}} \equiv \exp \left[\sum_{j=1}^n \int_0^{\infty} \overline{g_j(t)} f_j(t) dt \right]. \quad (3)$$

The normalized vectors

$$\iota(f) = \exp \left(-\frac{1}{2} \|f\|_{\mathcal{K}}^2 \right) e(f), \quad (4)$$

are called coherent vectors. In particular the coherent vector $\iota(0) = (1, 0, 0, \dots) \in \mathcal{F}$ is the vacuum state. The coherent state $\iota(f)$ can represent the laser light.

The linear span \mathcal{D} of all the exponential vectors is dense in \mathcal{F} . On the dense domain \mathcal{D} we define the annihilation $A_j(t)$, creation $A_j^\dagger(t)$ and number $A_{ij}(t)$ processes as follows [9, 11]:

$$A_j(t) e(f) = \int_0^t f_j(s) ds e(f), \quad (5)$$

$$A_j^\dagger(t) e(f) = \left. \frac{\partial}{\partial \epsilon_j} e(f + \epsilon \chi_{[0,t]}) \right|_{\epsilon=0}, \quad (6)$$

$$A_{ij}(t)e(f) = -i \frac{d}{d\lambda} e(\exp(i\lambda P_{ij} \chi_{[0,t]})f) \Big|_{\lambda=0}, \quad (7)$$

where $\chi_{[0,t]}$ is the indicator function of $[0, t]$, $\epsilon \equiv (\epsilon_1, \dots, \epsilon_n) \in \mathbb{R}^n$, $\lambda \in \mathbb{R}$ and $(P_{ij}f)_k = \delta_{ik}f_j$. The operators $A_j(t)$, $A_j^\dagger(t)$, $\Lambda_{ij}(t)$ satisfy the commutation relations of the form

$$\begin{aligned} [A_i(t), A_j(t')] &= [A_i^\dagger(t), A_j^\dagger(t')] = 0, \quad [\Lambda_{ij}(t), \Lambda_{kl}(t')] = \delta_{jk} \Lambda_{il}(t \wedge t') - \delta_{il} \Lambda_{kj}(t \wedge t'), \\ [A_i(t), A_j^\dagger(t')] &= \delta_{ij} t \wedge t', \quad [A_j(t), \Lambda_{kl}(t')] = \delta_{jk} \Lambda_{il}(t \wedge t'), \quad [\Lambda_{kl}(t), A_j^\dagger(t')] = \delta_{lj} \Lambda_{ik}^\dagger(t \wedge t'), \end{aligned} \quad (8)$$

where $t \wedge t' = \min(t, t')$.

The Fock space \mathcal{F} has a continuous tensor product structure, i.e.

$$\mathcal{F} = \mathcal{F}_{[0,t]} \otimes \mathcal{F}_{[t,\infty)}, \quad (9)$$

where $\mathcal{F}_{[0,t]}$ and $\mathcal{F}_{[t,\infty)}$ are the symmetric Fock spaces over $\mathbb{C}^n \otimes L^2([0, t])$ and $\mathbb{C}^n \otimes L^2([t, \infty))$, respectively. Let \mathcal{H} be a Hilbert space associated with some quantum system. The family $\{M(t), t \geq 0\}$ of operators on $\mathcal{H} \otimes \mathcal{F}$ is called a quantum adapted process if $M(t)$ acts as the identity operator in $\mathcal{F}_{[t,\infty)}$ and can act in non-trivial way in $\mathcal{H} \otimes \mathcal{F}_{[0,t]}$ [9].

Making use of the factorisation property (9), Hudson and Parthasarathy gave a rigorous meaning to the quantum stochastic equations (QSDE) of the type [9, 11]

$$dM(t) = \sum_{j=1}^n \left(\sum_{i=1}^n F_{ji}(t) d\Lambda_{ji}(t) + E_j(t) dA_j(t) + D_j(t) dA_j^\dagger(t) \right) + C(t)dt, \quad (10)$$

where $M(t)$, $F_{ji}(t)$, $E_j(t)$, $D_j(t)$, $C(t)$ are adapted processes on $\mathcal{H} \otimes \mathcal{F}$. The increments $dA_j(t)$, $dA_j^\dagger(t)$, $d\Lambda_{ij}(t)$ acting in $\mathcal{H} \otimes \mathcal{F}_{[t,t+dt]}$ commute with any adapted process $N(t)$ in $\mathcal{H} \otimes \mathcal{F}$.

To work with these equations we use the quantum Ito's rule which reads as follows. If $M'(t)$ is the process which satisfies an equation of the type (10), then the differential of the product $M(t)M'(t)$ is given by the formula [9, 11]

$$d(M(t)M'(t)) = dM(t)M'(t) + M(t)dM'(t) + dM(t)dM'(t), \quad (11)$$

where $dM(t)dM'(t)$ can be computed with the help of the multiplication table

$$\begin{aligned} dA_i(t) dA_j^\dagger(t) &= \delta_{ij} dt, \quad dA_i(t) d\Lambda_{kj}(t) = \delta_{ik} dA_j(t), \\ d\Lambda_{kj}(t) dA_i^\dagger(t) &= \delta_{ji} dA_k^\dagger(t), \quad d\Lambda_{ij}(t) d\Lambda_{kl}(t) = \delta_{jk} d\Lambda_{il}(t), \end{aligned} \quad (12)$$

and all other products vanish.

3. Filtering equation for photon counting measurement

Let us consider a single-mode field in a cavity (system \mathcal{S}) interacting with the Bose field. We assume that the unitary evolution operator $U(t)$ of this compound system satisfies the QSDE of the form

$$dU(t) = \left(\sqrt{\mu} b dA^\dagger(t) - \sqrt{\mu} b^\dagger dA(t) - \frac{\mu}{2} b^\dagger b dt - \frac{i}{\hbar} H dt \right) U(t), \quad U(0) = I, \quad (13)$$

where b stands for an annihilation operator, $H = \hbar\omega (b^\dagger b + \frac{1}{2})$ is the hamiltonian of \mathcal{S} and $\mu \in \mathbb{R}$ is a real coupling constant. The equation (13) is written in the interaction picture with respect to the free dynamics of the Bose field [17, 18]. The field described by the operators $A(t)$, $A^\dagger(t)$, and $\Lambda(t)$ is called the input field [10, 20] that is the field before the interaction with the system \mathcal{S} . The output processes, carrying the information about \mathcal{S} , are given by the formulae:

$$A^{\text{out}}(t) = U^\dagger(t) A(t) U(t), \quad \Lambda^{\text{out}}(t) = U^\dagger(t) \Lambda(t) U(t). \quad (14)$$

In this section we shall consider a direct detection of the output signal

$$\Lambda^{\text{out}}(t) = \int_0^t \left(d\Lambda(t') + \sqrt{\mu} b_{t'} dA^\dagger(t') + \sqrt{\mu} b_{t'}^\dagger dA(t') + \mu b_{t'}^\dagger b_{t'} dt' \right), \quad (15)$$

where $b_t = U^\dagger(t) b U(t)$. The formula (15) can be obtained from (14) and (13) by applying the rules of QSC. Note that the self-adjoint process $\Lambda^{\text{out}}(t)$ satisfies the non-demolition principle [1, 2]

$$[\Lambda^{\text{out}}(t'), U^\dagger(t) Z U(t)] = 0 \quad 0 \leq t' \leq t \quad (16)$$

for any observable Z of the system \mathcal{S} , and the commutativity condition

$$[\Lambda^{\text{out}}(t), \Lambda^{\text{out}}(t')] = 0 \quad \forall t, t' \geq 0. \quad (17)$$

Therefore $\Lambda^{\text{out}}(t)$ can be represented by a classical random measure on \mathbb{R} with values in $\{0, 1, 2, \dots\}$.

Let us suppose that one realizes a measurement of the output process (15) for the Bose field initially prepared in the coherent state $\iota(f)$. To get the filtering equation corresponding to the observation of a coherent channel, we introduce the generating map, $g(k, t)$, defined by

$$g(k, t) : Z \rightarrow g(k, t)[Z],$$

$$\langle \psi | g(k, t)[Z] \psi \rangle = \langle \psi \otimes \iota(f) | G^{\text{out}}(k, t) Z_t \psi \otimes \iota(f) \rangle, \quad (18)$$

where

$$G^{\text{out}}(k, t) = \exp \left\{ \int_0^t \ln k(t') d\Lambda^{\text{out}}(t') \right\}, \quad (19)$$

$Z_t = U^\dagger(t)ZU(t)$ is the Heisenberg operator of \mathcal{S} , ψ stands for the initial state of \mathcal{S} , and k is a complex measurable test function satisfying $0 < |k| \leq 1$.

An explicit expression for the generating map (18) can be obtained by solving the differential equation for $g(k, t)$. In order to find this equation, one should first obtain the QSDE for the operator $\pi_k^{\text{out}}(t, Z) = U^\dagger(t)G(k, t)ZU(t)$, where $G(k, t) = U(t)G^{\text{out}}(k, t)U^\dagger(t)$. By the rules of QSC, one can easily check that

$$dG(k, t) = (k(t) - 1) d\Lambda(t)G(k, t). \quad (20)$$

From (13) and (20), one obtains

$$\begin{aligned} d\pi_k^{\text{out}}(t, Z) = & -(K_t^\dagger \pi_k^{\text{out}}(t, Z) + \pi_k^{\text{out}}(t, Z)K_t)dt + k(t) \mu b_t^\dagger \pi_k^{\text{out}}(t, Z)b_t dt \\ & + \sqrt{\mu}(b_t^\dagger \pi_k^{\text{out}}(t, Z) - \pi_k^{\text{out}}(t, Z)b_t^\dagger)dA(t) + \sqrt{\mu}(\pi_k^{\text{out}}(t, Z)b_t - b_t \pi_k^{\text{out}}(t, Z))dA^\dagger(t) \\ & + (k(t) - 1)(\pi_k^{\text{out}}(t, Z)d\Lambda(t) + \sqrt{\mu}b_t^\dagger \pi_k^{\text{out}}(t, Z)dA(t) + \sqrt{\mu}\pi_k^{\text{out}}(t, Z)b_t dA^\dagger(t)), \end{aligned} \quad (21)$$

where $K_t = U^\dagger(t)KU(t)$ and $K = \frac{i}{\hbar}H + \frac{\mu}{2}b^\dagger b$.

By taking the mean value of both sides of the equation (21) with respect to the state $\eta = \psi \otimes \iota(f)$, one gets the differential equation for the mean value of $\pi_k^{\text{out}}(t, Z)$:

$$\begin{aligned} \langle d\pi_k^{\text{out}}(t, Z) \rangle = & \langle \eta(t) | - (K^\dagger \pi_k(t, Z) + \pi_k(t, Z)K) + k(t) \mu b^\dagger \pi_k(t, Z)b \\ & + \sqrt{\mu}(b^\dagger \pi_k(t, Z) - \pi_k(t, Z)b^\dagger)f(t) + \sqrt{\mu}(\pi_k(t, Z)b - b \pi_k(t, Z))\overline{f(t)} \\ & + (k(t) - 1)(\pi_k(t, Z)|f(t)|^2 + \sqrt{\mu}b^\dagger \pi_k(t, Z)f(t) + \sqrt{\mu}\pi_k(t, Z)b\overline{f(t)}) | \eta(t) \rangle dt, \end{aligned} \quad (22)$$

where $\pi_k(t, Z) = G(k, t)Z$ and $\eta(t) = U(t)\eta$. Hence, the generating map $g(k, t)$ satisfies the differential equation

$$\begin{aligned} \frac{d}{dt}g(k, t)[Z] = & g(k, t)[-K^\dagger Z - ZK - \sqrt{\mu}(Zb^\dagger f(t) + bZ\overline{f(t)}) - Z|f(t)|^2 \\ & + k(t)(\mu b^\dagger Zb + \sqrt{\mu}b^\dagger Zf(t) + \sqrt{\mu}Zb\overline{f(t)} + Z|f(t)|^2)] \end{aligned} \quad (23)$$

with the initial condition $g(k, 0)[Z] = Z$.

The solution to eq. (23) is given by the von Neumann-Dyson series

$$g(k, t)[Z] = \sum_{n=0}^{\infty} \int_0^t dt_n \int_0^{t_n} dt_{n-1} \dots \int_0^{t_2} dt_1 k(t_1) \dots k(t_n) \times \quad (24)$$

$$\times S^\dagger(t_1) \dots S^\dagger(t_n)Z(t)S(t_n) \dots S(t_1), \quad (25)$$

where

$$Z(t) = e^{-L^\dagger(t)}Z e^{-L(t)}, \quad (26)$$

$$S(t) = e^{L(t)}\left(\sqrt{\mu}b + f(t)\right)e^{-L(t)}, \quad (27)$$

with

$$L(t) = Kt + \int_0^t \left(\sqrt{\mu} b^\dagger f(t') + \frac{|f(t')|^2}{2} \right) dt'. \quad (28)$$

Let $\tau = (t_1, t_2, \dots, t_n)$ be the trajectory of the observed counting process $\Lambda^{\text{out}}(t)$ up to t and Σ^t is a set of all finite chains $|\tau| = n \in \{0, 1, 2, \dots\}$. By introducing the stochastic operator $V(\tau | t) = e^{-L(t)} S(t_n) \dots S(t_1)$, one can rewrite the series (24) in the form

$$g(k, t)[Z] = \int_{\tau \in \Sigma^t} k(\tau) V^\dagger(\tau | t) Z V(\tau | t) d\tau, \quad (29)$$

where $k(\tau) = \prod_{i=1}^n k(t_i)$ and $d\tau = \prod_{i=1}^n dt_i$. The stochastic propagator $\widehat{V}(t)(\tau) = V(\tau | t)$ defining for any trajectory τ the posterior state $\widehat{\psi}(t)(\tau) = V(\tau | t)\psi$ of \mathcal{S} , can be represented by the Ito chronological multiplicative integral

$$\widehat{V}(t) = e^{-L(t)} \sum_{n=0}^{\infty} \int_0^t dt_n \int_0^{t_n} dt_{n-1} \dots \int_0^{t_2} dt_1 S'(t_n) \dots S'(t_1) \prod_{i=1}^n dN(t_i). \quad (30)$$

In the last formula

$$S'(t) = e^{L(t)} \left(\sqrt{\mu} b + f(t) - I \right) e^{-L(t)} \quad (31)$$

and $N(t)$ is a random variable such that for $\tau \in \Sigma^\infty$ one has $dN(t)(\tau) = 1$ if $t \in \tau$ and $dN(t)(\tau) = 0$ if $t \notin \tau$.

By differentiation of (30), we get the linear stochastic equation

$$d\widehat{V}(t) = - \left(K + \sqrt{\mu} b^\dagger f(t) + \frac{|f(t)|^2}{2} \right) \widehat{V}(t) dt + \left(\sqrt{\mu} b + f(t) - I \right) \widehat{V}(t) dN(t), \quad \widehat{V}(0) = I. \quad (32)$$

Thus, we draw the conclusion that the posterior unnormalized wave function $\widehat{\psi}(t) = \widehat{V}(t)\psi$ satisfies the linear Belavkin equation of the form

$$d\widehat{\psi}(t) = - \left(K + \sqrt{\mu} b^\dagger f(t) + \frac{|f(t)|^2}{2} \right) \widehat{\psi}(t) dt + \left(\sqrt{\mu} b + f(t) - I \right) \widehat{\psi}(t) dN(t), \quad \widehat{\psi}(0) = \psi. \quad (33)$$

The nonlinear filtering equation preserving the normalization of the posterior wave function can be calculated from eq. (33). To derive the differential equation for $\widehat{\varphi}(t) = \langle \widehat{\psi}(t) | \widehat{\psi}(t) \rangle^{-1/2} \widehat{\psi}(t)$ one has to check that

$$\begin{aligned} d \left(\langle \widehat{\psi}(t) | \widehat{\psi}(t) \rangle \right) &= - \langle \widehat{\psi}(t) | \left(\sqrt{\mu} b^\dagger + \overline{f(t)} \right) \left(\sqrt{\mu} b + f(t) \right) | \widehat{\psi}(t) \rangle dt \\ &\quad + \langle \widehat{\psi}(t) | \left[\left(\sqrt{\mu} b^\dagger + \overline{f(t)} \right) \left(\sqrt{\mu} b + f(t) \right) - 1 \right] | \widehat{\psi}(t) \rangle dN(t), \end{aligned} \quad (34)$$

and insert it into the Taylor expansion of $d(\langle\hat{\psi}(t)|\hat{\psi}(t)\rangle^{-1/2})$ which yields

$$\begin{aligned} d(\langle\hat{\psi}(t)|\hat{\psi}(t)\rangle^{-1/2}) &= \\ &= \langle\hat{\psi}(t)|\hat{\psi}(t)\rangle^{-1/2} \left\{ \frac{1}{2} \left(\mu\langle b^\dagger b \rangle_t + 2\sqrt{\mu} \operatorname{Re}(\langle b \rangle_t \overline{f(t)}) + |f(t)|^2 \right) dt \right. \\ &\quad \left. + \left[\left(\mu\langle b^\dagger b \rangle_t + 2\sqrt{\mu} \operatorname{Re}(\langle b \rangle_t \overline{f(t)}) + |f(t)|^2 \right)^{-1/2} - 1 \right] dN(t) \right\}, \end{aligned} \quad (35)$$

where $\langle \cdot \rangle_t = \langle \hat{\varphi}(t) | (\cdot) | \hat{\varphi}(t) \rangle$. By eqs. (33), (35) and the Ito formula

$$d\hat{\varphi}(t) = d \left(\langle\hat{\psi}(t)|\hat{\psi}(t)\rangle^{-1/2} \right) \hat{\psi}(t) + \langle\hat{\psi}(t)|\hat{\psi}(t)\rangle^{-1/2} d\hat{\psi}(t) + d \left(\langle\hat{\psi}(t)|\hat{\psi}(t)\rangle^{-1/2} \right) d\hat{\psi}(t) \quad (36)$$

we obtain the nonlinear filtering equation of the form

$$\begin{aligned} d\hat{\varphi}(t) &= \left(-K - \sqrt{\mu} b^\dagger f(t) + \mu/2 \langle b^\dagger b \rangle_t + \sqrt{\mu} \operatorname{Re}(\langle b \rangle_t \overline{f(t)}) \right) \hat{\varphi}(t) dt \\ &\quad + \left(\frac{\sqrt{\mu} b + f(t)}{\sqrt{\mu\langle b^\dagger b \rangle_t + 2\sqrt{\mu} \operatorname{Re}(\langle b \rangle_t \overline{f(t)}) + |f(t)|^2}} - I \right) \hat{\varphi}(t) dN(t), \quad \hat{\varphi}(0) = \psi \end{aligned} \quad (37)$$

If the initial state of the system \mathcal{S} is given by a density matrix ρ , then the posterior normalized density matrix $\hat{\rho}(t)$ satisfies the stochastic equation

$$\begin{aligned} d\hat{\rho}(t) &= -\frac{i}{\hbar} [H, \hat{\rho}(t)] dt - \frac{\mu}{2} \{b^\dagger b, \hat{\rho}(t)\} dt + \mu b \hat{\rho}(t) b^\dagger dt + \sqrt{\mu} [b \overline{f(t)} - b^\dagger f(t), \hat{\rho}(t)] dt \\ &\quad + \left(\frac{(\sqrt{\mu} b + f(t)) \hat{\rho}(t) (\sqrt{\mu} b^\dagger + \overline{f(t)})}{\mu\langle b^\dagger b \rangle_t + 2\sqrt{\mu} \operatorname{Re}(\langle b \rangle_t \overline{f(t)}) + |f(t)|^2} - \hat{\rho}(t) \right) \\ &\quad \times \left(dN(t) - \mu\langle b^\dagger b \rangle_t dt - 2\sqrt{\mu} \operatorname{Re}(\langle b \rangle_t \overline{f(t)}) dt - |f(t)|^2 dt \right), \end{aligned} \quad (38)$$

where $\{c, d\} = cd + dc$. The validity of the above equation can be checked by differentiating $\hat{\rho}(t) = |\hat{\varphi}(t)\rangle\langle\hat{\varphi}(t)|$.

By taking the stochastic mean of (38), we obtain the master equation

$$\frac{d}{dt} \sigma(t) = -\frac{i}{\hbar} [H, \sigma(t)] - \frac{\mu}{2} \{b^\dagger b, \sigma(t)\} + \mu b \sigma(t) b^\dagger + \sqrt{\mu} [b \overline{f(t)} - b^\dagger f(t), \sigma(t)] \quad (39)$$

for the prior state

$$\sigma(t) = \langle \hat{\rho}(t) \rangle_{st}, \quad (40)$$

since the posterior mean value of $dN(t)$ is given by

$$\langle dN(t) \rangle(\tau) = \mu\langle b^\dagger b \rangle_t dt + 2\sqrt{\mu} \operatorname{Re}(\langle b \rangle_t \overline{f(t)}) dt + |f(t)|^2 dt. \quad (41)$$

Let us note that for $f(t) = 0$ the equation (38) takes the form of a quantum filtering equation for the reservoir prepared initially in the vacuum state, derived, for instance, in [1].

In the model worked out by Hudson and Parthasarathy, the range of frequency of the reservoir extends from $-\infty$ to $+\infty$. A coherent state of the continuum mode distribution, having a very narrow spectral width, is here only an analogue of a single-mode laser field and $f(t) = \lambda \exp(-i\omega t + i\phi)$, $\lambda \geq 0$ [21]. In the considered experiment the laser light constitutes a coherent signal which stimulates the system \mathcal{S} and reaches the detector. We observe the interference between the laser light and the light emitted by the system \mathcal{S} in the same channel. Because of the presence of the terms corresponding to the stimulation of the system \mathcal{S} , we cannot take the limit $|f| \rightarrow \infty$ in (38).

4. Heterodyne measurement. Transition from counting to diffusion process

In the heterodyne detection scheme, depicted in Fig. 1, the measured signal is superposed with an auxiliary laser field (local oscillator) by means of the beam splitter [17, 18, 21]. For a lossless beam splitter of transmissivity T , the field reaching detector can be written as

$$A_{\text{mix}}(t) = \sqrt{T}A^{\text{out}}(t) + i\sqrt{1-T}A_{\text{lo}}(t). \quad (42)$$

The operators $A_{\text{lo}}(t)$, $A_{\text{lo}}^\dagger(t)$ represent the local oscillator and satisfy the commutation relations (8). The auxiliary field does not interact with \mathcal{S} and we assume that its initial state is given by the coherent vector $\iota(f_{\text{lo}})$.

The output generating operator for the ordinary heterodyne measurement is defined as

$$G^{\text{out}}(k, t) = \langle \iota(f_{\text{lo}}) | \exp \left\{ \int_0^t \ln k(t') dA_{\text{mix}}(t') \dot{A}_{\text{mix}}(t') \right\} \iota(f_{\text{lo}}) \rangle, \quad (43)$$

where $\dot{A}_{\text{mix}}(t) = dA_{\text{mix}}(t)/dt$. In this way the degrees of freedom of the auxiliary field have been eliminated from our description. One can check using (11) and (12) that $(dA_{\text{mix}}(t)\dot{A}_{\text{mix}}(t))^2 = dA_{\text{mix}}(t)\dot{A}_{\text{mix}}(t)$. The assumptions that the field $A^{\text{out}}(t)$ is not lost, i.e. $T \rightarrow 1$, together with $|f_{\text{lo}}| \rightarrow \infty$, such that the product $(1-T)|f_{\text{lo}}|^2 := \varepsilon^{-2}$ is fixed [10, 16], lead to the formula

$$G^{\text{out}}(k, t) = \exp \left\{ \int_0^t \ln k(t') dY^{\text{out}}(t') \right\} \quad (44)$$

with the output counting process $Y^{\text{out}}(t)$ of the form

$$Y^{\text{out}}(t) = \int_0^t dA^{\text{out}}(t') + \frac{r(t')}{\varepsilon} dA^{\text{out}\dagger}(t') + \frac{\overline{r(t')}}{\varepsilon} dA^{\text{out}}(t') + \frac{1}{\varepsilon^2} dt', \quad (45)$$

$r(t)$ is a complex function with modulus $|r(t)| = 1$.

Let us note that from the relations (8) it follows that $Y(t) = U(t)Y^{\text{out}}(t)U^\dagger(t)$ commutes with $Y(t')$ for any t and t' . Hence, taking into account that [20] $U(t)Y^{\text{out}}(t') = Y(t')U(t)$ one gets

$$[Y^{\text{out}}(t), Y^{\text{out}}(t')] = 0 \quad \forall t, t' \geq 0. \quad (46)$$

The property (46) allows us to treat the process (45) as a classical one.

The generating map $g(k, t)$ for the operator $G^{\text{out}}(k, t)$ defined by (44) satisfies the differential equation

$$\begin{aligned} \frac{d}{dt}g(k, t)[Z] = & g(k, t) \left[- \left(K + \sqrt{\mu} b^\dagger f(t) + \sqrt{\mu} b \overline{r(t)}/\varepsilon + 1/2 |f(t) + r(t)/\varepsilon|^2 \right)^\dagger Z \right. \\ & - Z \left(K + \sqrt{\mu} b^\dagger f(t) + \sqrt{\mu} b \overline{r(t)}/\varepsilon + 1/2 |f(t) + r(t)/\varepsilon|^2 \right) \\ & \left. + k(t) \left(\sqrt{\mu} b^\dagger + \overline{f(t)} + \overline{r(t)}/\varepsilon \right) Z \left(\sqrt{\mu} b + f(t) + r(t)/\varepsilon \right) \right] \end{aligned} \quad (47)$$

with $g(k, 0)[Z] = Z$. The solution to this equation can be written in the form

$$g(k, t)[Z] = \int_{\kappa \in \Omega^t} k(\kappa) V^\dagger(\kappa | t) Z V(\kappa | t) d\kappa, \quad (48)$$

where κ denotes the counting trajectory of registered photons for heterodyne measurement, $\Omega^t = \bigcup_{n=0}^{\infty} \{\kappa \subset [0, t) : |\kappa| = n\}$, and

$$V(\kappa | t) = e^{-R(t)} S(t_n) \dots S(t_1), \quad (49)$$

where

$$S(t) = e^{R(t)} \left(\sqrt{\mu} b + f(t) + \frac{r(t)}{\varepsilon} \right) e^{-R(t)}, \quad (50)$$

$$R(t) = Kt + \int_0^t \left(\sqrt{\mu} b^\dagger f(t') + \sqrt{\mu} b \overline{r(t')}/\varepsilon + 1/2 |f(t') + r(t')/\varepsilon|^2 \right) dt'. \quad (51)$$

Here, the stochastic propagator $\widehat{V}(t)(\kappa) = V(\kappa | t)$ given by the formula

$$\widehat{V}(t) = e^{-R(t)} \sum_{n=0}^{\infty} \int_0^t dt_n \int_0^{t_n} dt_{n-1} \dots \int_0^{t_2} dt_1 S'(t_n) \dots S'(t_1) \prod_{i=1}^n dY(t_i), \quad (52)$$

where

$$S'(t) = e^{R(t)} \left(\sqrt{\mu} b + f(t) + \frac{r(t)}{\varepsilon} - I \right) e^{-R(t)}, \quad (53)$$

and $Y(t)$ is a random variable such that $dY(t)(\kappa) = 1$ for $t \in \kappa$ and $dY(t)(\kappa) = 0$ for $t \notin \kappa$, satisfies the equation

$$d\widehat{V}(t) = -R(t)\widehat{V}(t)dt + \left(\sqrt{\mu} b + f(t) + \frac{r(t)}{\varepsilon} - I \right) \widehat{V}(t)dY(t), \quad \widehat{V}(0) = I. \quad (54)$$

Hence, we get for the posterior unnormalized wave function $\widehat{\psi}(t) = \widehat{V}(t)\psi$ the linear Belavkin equation of the form

$$d\widehat{\psi}(t) = -R(t)\widehat{\psi}(t)dt + \left(\sqrt{\mu}b + f(t) + \frac{r(t)}{\varepsilon} - I\right)\widehat{\psi}(t)dY(t), \quad \widehat{\psi}(0) = \psi. \quad (55)$$

From the equation (55) one can easily derive the nonlinear filtering equation for the normalized vector $\widehat{\varphi}(t) = \langle \widehat{\psi}(t) | \widehat{\psi}(t) \rangle^{-1/2} \widehat{\psi}(t)$,

$$\begin{aligned} d\widehat{\varphi}(t) = & - \left(K + \sqrt{\mu}b^\dagger f(t) + \sqrt{\mu}b\overline{r(t)}/\varepsilon \right) \widehat{\varphi}(t)dt \\ & + \left(\mu/2 \langle b^\dagger b \rangle_t + \sqrt{\mu} \operatorname{Re} \left(\langle b \rangle_t \left(\overline{f(t)} + \overline{r(t)}/\varepsilon \right) \right) \right) \widehat{\varphi}(t)dt \\ & + \left(\left(\mu \langle b^\dagger b \rangle_t + 2\sqrt{\mu} \operatorname{Re} \left(\langle b \rangle_t \left(\overline{f(t)} + \overline{r(t)}/\varepsilon \right) \right) + |f(t) + r(t)/\varepsilon|^2 \right)^{-1/2} \right. \\ & \quad \left. \times (\sqrt{\mu}b + f(t) + r(t)/\varepsilon - I) \widehat{\varphi}(t) dY(t), \right. \end{aligned} \quad (56)$$

where $\langle . \rangle_t = \langle \widehat{\varphi}(t) | (.) \widehat{\varphi}(t) \rangle$ is the posterior mean value of an operator of the system \mathcal{S} . Furthermore, the filtering equation for the posterior density matrix corresponding to (56) has the form

$$\begin{aligned} d\widehat{\rho}(t) = & \left(-\frac{i}{\hbar} [H, \widehat{\rho}(t)] - \frac{\mu}{2} \{b^\dagger b, \widehat{\rho}(t)\} + \mu b \widehat{\rho}(t) b^\dagger + \sqrt{\mu} [b\overline{f(t)} - b^\dagger f(t), \widehat{\rho}(t)] \right) dt \\ & + \left(\varepsilon \mu b \widehat{\rho}(t) b^\dagger + \sqrt{\mu} b \left(\varepsilon \overline{f(t)} + \overline{r(t)} \right) \widehat{\rho}(t) + \widehat{\rho}(t) \sqrt{\mu} b^\dagger (\varepsilon f(t) + r(t)) \right. \\ & \quad \left. - \varepsilon \mu \langle b^\dagger b \rangle_t \widehat{\rho}(t) - 2\sqrt{\mu} \operatorname{Re} \left(\langle b \rangle_t \left(\varepsilon \overline{f(t)} + \overline{r(t)} \right) \right) \widehat{\rho}(t) \right) \\ & \times \left(\varepsilon dY(t) - \varepsilon \mu \langle b^\dagger b \rangle_t dt - 2\sqrt{\mu} \operatorname{Re} \left(\langle b \rangle_t \left(\varepsilon \overline{f(t)} + \overline{r(t)} \right) \right) dt - \varepsilon^{-1} |\varepsilon f(t) + r(t)|^2 dt \right) \\ & \times \left(\varepsilon^2 \mu \langle b^\dagger b \rangle_t + 2\sqrt{\mu} \operatorname{Re} \left(\langle b \rangle_t \left(\varepsilon^2 \overline{f(t)} + \varepsilon \overline{r(t)} \right) \right) + |\varepsilon f(t) + r(t)|^2 \right)^{-1}. \end{aligned} \quad (57)$$

Of course, if we put $r(t) = 0$ in the above equations, we get the results obtained in the previous section for a direct observation of the output field. Moreover, since the posterior mean value of $dY(t)$ is given by the formula

$$\langle dY(t) \rangle(\kappa) = \left(\mu \langle b^\dagger b \rangle_t + 2\sqrt{\mu} \varepsilon^{-1} \operatorname{Re} \left(\langle b \rangle_t \left(\varepsilon \overline{f(t)} + \overline{r(t)} \right) \right) + \varepsilon^{-2} |\varepsilon f(t) + r(t)|^2 \right) dt, \quad (58)$$

the averaging of (57) on the past leads to the master equation of the form (39).

Now we can consider the linear transformation of the counting process $Y(t)$ [3],

$$dW^\varepsilon(t) = \varepsilon dY(t) - \frac{dt}{\varepsilon}, \quad (59)$$

leading in the limit $\varepsilon \rightarrow 0$ to diffusion observation. One can easily check that

$$dW^\varepsilon(t) dW^\varepsilon(t) = \varepsilon^2 dY(t) = \varepsilon dW^\varepsilon(t) + dt, \quad dW^\varepsilon(t) dt = 0, \quad (60)$$

such that for $W(t) = \lim_{\varepsilon \rightarrow 0} W^\varepsilon(t)$ we obtain the following Ito rules:

$$dW(t)dW(t) = dt, \quad dW(t)dt = 0. \quad (61)$$

Taking $\varepsilon \rightarrow 0$ one can get from (57) the filtering equation for the diffusion observation

$$\begin{aligned} d\hat{\rho}(t) = & \left(-\frac{i}{\hbar}[H, \hat{\rho}(t)] - \frac{\mu}{2}\{b^\dagger b, \hat{\rho}(t)\} + \mu b \hat{\rho}(t) b^\dagger + \sqrt{\mu}[b \overline{f(t)} - b^\dagger f(t), \hat{\rho}(t)] \right) dt \\ & + \left(\sqrt{\mu} \overline{r(t)} (b - \langle b \rangle_t) \hat{\rho}(t) + \sqrt{\mu} r(t) \hat{\rho}(t) (b^\dagger - \langle b^\dagger \rangle_t) \right) \\ & \times \left(dW(t) - 2 \operatorname{Re} \left(\sqrt{\mu} \langle b \rangle_t \overline{r(t)} + f(t) \overline{r(t)} \right) dt \right). \end{aligned} \quad (62)$$

Let us note that eq. (62) transforms pure states into pure ones. In such a case, for $\hat{\rho}(t) = |\hat{\varphi}(t)\rangle\langle\hat{\varphi}(t)|$ eq. (62) takes the form

$$\begin{aligned} d\hat{\varphi}(t) = & \left(-K + \sqrt{\mu} b \overline{f(t)} - \sqrt{\mu} b^\dagger f(t) + \mu \langle b^\dagger \rangle_t b - \frac{\mu}{2} |\langle b \rangle_t|^2 \right) \hat{\varphi}(t) dt \\ & + \sqrt{\mu} \overline{r(t)} (b - \langle b \rangle_t) \hat{\varphi}(t) \left(dW(t) - 2 \operatorname{Re} \left(\sqrt{\mu} \langle b \rangle_t \overline{r(t)} + f(t) \overline{r(t)} \right) dt \right). \end{aligned} \quad (63)$$

Let us note that if we take $f(t) = 0$, then all the obtained formulae reduce to the known results for the reservoir initially prepared in the vacuum state [3]. Finally it is worth to mention that modelling of the evolution of a quantum state conditioned by the results of a continuous measurement is the first step towards quantum feedback control [6, 8] with a coherent source used as a control field [22]. As the choice of the system's operators does not interfere with the derivation of the filtering equations for a measurement of a coherent channel, the results of the paper can be easily generalized – in the obtained filtering equations, the system's hamiltonian and system's coupling operator can be replaced with arbitrary ones.

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Figure captions

Fig. 1. The heterodyne setup

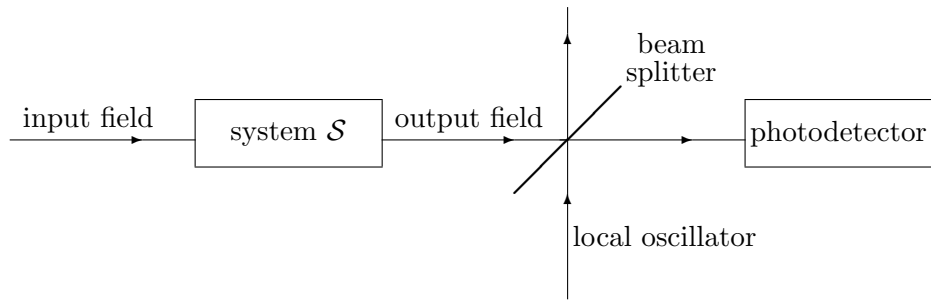


Fig. 1. The heterodyne setup